VII. Second order equations, complex roots

Lesson Overview

- To solve the (linear, second-order, homogeneous, constant coefficient) differential equation

$$ay'' + by' + cy = 0$$

first solve the characteristic equation:

$$ar^2 + br + c = 0$$

Lesson Overview

- Sometimes the characteristic equation has complex roots $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$. (They always come in conjugate pairs if the original equation had real coefficients.) Then the general solution to the differential equation is

$$\boxed{y_{\text{gen}} = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t}.$$

- As before, to find c_1 and c_2, use initial conditions, usually given as $y(0)$ and $y'(0)$. You'll get two equations in two unknowns.

Example I

Find the general solution to the differential equation:

$$y'' - 4y' + 13y = 0$$

$r = 2 \pm 3i \implies$

General Solution: $\boxed{y_{\text{gen}} = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t}$.

Example II

Solve the initial value problem:

$$y'' - 4y' + 13y = 0, y(0) = 2, y'(0) = 1$$

General Solution from before:

$y_{\text{gen}} = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t \implies y(0) = c_1 = 2$

$y' = 2c_1 e^{2t} \cos 3t - 3c_1 e^{2t} \sin 3t + 2c_2 e^{2t} \sin 3t + 3c_2 e^{2t} \cos 3t \implies y'(0) = 2c_1 + 3c_2 = 1$

$2(2) + 3c_2 = 1 \implies c_2 = -1$

$y = \boxed{2e^{2t} \cos 3t - e^{2t} \sin 3t}$

Example III

Find the general solution of $y'' - 10y' + 29y = 0$.

$r = 5 \pm 2i \implies \boxed{y_{\text{gen}} = c_1 e^{5t} \cos 2t + c_2 e^{5t} \sin 2t}$

Example IV

Solve the initial value problem:

$$y'' + 4y = 0, y(0) = 5, y'(0) = 6$$

Be careful not to write $r^2 + 4r = 0$.

$$\begin{aligned} r^2 + 4 &= 0 \Longrightarrow r = 0 \pm 2i \\ y_{\text{gen}} &= c_1 \cos 2t + c_2 \sin 2t \Longrightarrow y(0) = c_1 = 5 \\ y' &= -2c_1 \sin 2t + 2c_2 \cos 2t \Longrightarrow y'(0) = 2c_2 = 6 \Longrightarrow c_2 = 3 \\ y &= \boxed{5 \cos 2t + 3 \sin 2t} \end{aligned}$$

Example V

Find the general solution of $y'' - 8y' + 20y = 0$.

$r = 4 \pm 2i \Longrightarrow \boxed{y_{\text{gen}} = c_1 e^{4t} \cos 2t + c_2 e^{4t} \sin 2t}$