VI. Bayes' Rule

When to use Bayes' Rule

- Your sample space must be a <u>disjoint</u> union of events:

$$S = B_1 \cup B_2 \cup \cdots \cup B_n$$

- Then you have one event A that overlaps the others.
- "Given that A occured, what is the probability that one of the B's is true?"

Draw Venn diagram.

Bayes' Rule for two choices

- If $S = B_1 \cup B_2$ is a <u>disjoint</u> union, then

$$P(B_1|A) = \frac{P(A|B_1)\,P(B_1)}{P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)}.$$

Bayes' Rule for multiple choices

- If $S = B_1 \cup \cdots \cup B_n$ is a <u>disjoint</u> union, then

$$\boxed{P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_{i=1}^{n} P(A|B_i)\,P(B_i)}}.$$

Example I

20% of people who take home diabetes tests actually are diabetic. The accuracy of one brand of test is as follows: If the subject is diabetic, there is a 90% chance that the test will show positive. If she is not, there is an 80% chance that the test will show negative. If a woman takes the test and it shows positive, what is the chance that she actually is diabetic?

Example I

Define

$$\begin{aligned} A &:= \text{test is positive.} \\ B_1 &:= \text{diabetic} \\ B_2 &:= \text{not diabetic} \end{aligned}$$

We want $P(B_1|A)$.

$$\begin{aligned} & P(\text{diabetic}|\text{positive}) \\ &= \frac{P(\text{positive}|\text{diabetic})P(\text{diabetic})}{P(\text{positive}|\text{diabetic})P(\text{diabetic}) + P(\text{positive}|\text{not diabetic})P(\text{not diabetic})} \\ P(B_1|A) &= \frac{P(A|B_1)\,P(B_1)}{P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)} \\ &= \frac{(0.9)(0.2)}{(0.9)(0.2) + (0.2)(0.8)} \\ &= \boxed{\frac{9}{17}} \end{aligned}$$

(Note that this is only about $\frac{1}{2}$, surprising considering that the test is accurate at least 80% of the time!)

Example II

A small college has 220 women and 150 men. 40% of the women play tennis, and 30% of the men do. If you find an extra tennis racket on the court, what is the chance that it belongs to a woman?

$$\begin{aligned}
P(\text{woman}|\text{tennis}) &= \frac{P(\text{tennis}|\text{woman})P(\text{woman})}{P(\text{tennis}|\text{woman})P(\text{woman}) + P(\text{tennis}|\text{man})P(\text{man})} \\
&= \frac{0.4 \cdot \frac{22}{37}}{0.4 \cdot \frac{22}{37} + 0.3 \cdot \frac{15}{37}} \\
&= \frac{0.4 \cdot 22}{0.4 \cdot 22 + 0.3 \cdot 15} \\
&\approx \boxed{0.662}
\end{aligned}$$

Example III

In the state of New Alarkania, 40% of the voters are Democrats, 50% are Republicans, and 10% are Freemasons. A survey shows that 20% of Democrats, 30% of Republicans, and 50% of Freemasons support freeway lanes for pogo sticks. A voter chosen at random supports the lanes. What is the probability that she is a Democrat?

Example III

Define the events:

$$\begin{aligned} A &:= \{\text{she supports the lanes}\} \\ B_D &:= \{\text{she is a Democrat}\} \\ B_R &:= \{\text{she is a Republican}\} \\ B_F &:= \{\text{she is Freemason}\}. \end{aligned}$$

$$\boxed{P\left(B_j|A\right) = \frac{P\left(A|B_j\right)P\left(B_j\right)}{\sum_{i=1}^{n} P\left(A|B_i\right)P\left(B_i\right)}}.$$

$$\begin{aligned} P\left(B_D|A\right) &= \frac{P(A|B_D)P(B_D)}{P(A|B_D)P(B_D) + P(A|B_R)P(B_R) + P(A|B_F)P(B_F)} \\ &= \frac{\frac{1}{5}\frac{2}{5}}{\frac{1}{5}\frac{2}{5} + \frac{3}{10}\frac{1}{2} + \frac{1}{2}\frac{1}{10}} \\ &= \frac{\frac{2}{25}}{\frac{7}{25}} = \boxed{\frac{2}{7}} \end{aligned}$$

Example IV

We have three bags, one with two apples, one with two oranges, and one with an apple and an orange. We draw one fruit from one bag, and it's an apple. What is the chance that the other fruit in the same bag is an apple?

Example IV

Events: $B_1 := B_{\text{aa}} :=$ two apples. $B_2 := B_{\text{oo}} :=$ two oranges. $B_3 := B_{\text{oa}} :=$ one of each.
$A :=$ we pull out an apple.
We are calculating $P(B_{\text{aa}}|A)$.

$$\boxed{P\left(B_j|A\right) = \frac{P\left(A|B_j\right)P\left(B_j\right)}{\sum_{i=1}^{n} P\left(A|B_i\right)P\left(B_i\right)}}.$$

$$\begin{aligned}
P(B_{\text{aa}}|A) &= \frac{P(A|B_{\text{aa}})P(B_{\text{aa}})}{P(B_{\text{aa}})P(A|B_{\text{aa}}) + P(B_{\text{oo}})P(A|B_{\text{oo}}) + P(B_{\text{oa}})P(A|B_{\text{oa}})} \\
&= \frac{1 \cdot \frac{1}{3}}{\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2}} \\
&= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}
\end{aligned}$$

Example V

A family has three children (with no twins). If there are at least two girls, what is the probability that the oldest child is a girl?

Bayes Solution:
Define

$$\begin{aligned} A &:= \text{at least two girls.} \\ B_1 &:= \text{oldest child is girl.} \\ B_2 &:= \text{oldest child is boy.} \end{aligned}$$

$$\begin{aligned} P(B_1|A) &= \frac{P(A|B_1)\,P(B_1)}{P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)} \\ &= \frac{\frac{3}{4}\cdot\frac{1}{2}}{\frac{3}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}} \\ &= \frac{\frac{3}{8}}{\frac{3}{8}+\frac{1}{8}} = \frac{\frac{3}{8}}{\frac{1}{2}} = \boxed{\frac{3}{4}} \end{aligned}$$

Counting solution: Our possibilities are GGG, GGB, GBG, and BGG. Three of them qualify, so it's $\boxed{\frac{3}{4}}$.